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Optimal Station-Keeping at Collinear Points

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An optimal control problem is formulated for stability control at the collinear libration points in the restricted three-body problem. A simple control is derived that can be applied intermittently to establish certain quasi-elliptic orbits around these points predicted by the linear analysis. Numerical simulation results for the Earth-moon system indicate that stability around the collinear points can be maintained for long periods of time using short corrective thrust applications with small fuel consumption.

Introduction

THE restricted three-body problem has been of interest to applied mathematicians for centuries and, in recent years, free trajectories for the restricted three-body model have received considerable analytical and numerical study. The particular locations of the singular or libration points in this system have suggested a number of interesting possibilities for scientific applications.[1–3] Among them are communications, radio astronomy, meteoroid collection, and manned space stations for trips to the moon.

Many studies have been made relating to the periodic orbits around the equilateral points and the sun's perturbation on these motions to see if bounded motion will persist.[4,5] Reference 14 considers the control problem of reaching the librations with zero relative speed. Specific control laws for station-keeping have received little attention because the specific utility of such satellite placements has not yet been defined. For the collinear libration points, however, a much more basic problem is present, that of the instability of small motions around the points. In Refs. 6 and 7 it is shown that all small perturbations away from the collinear points lead to instability in the sense of Liapunov. Therefore, a control effort will be necessary to keep any space probe in the vicinity of the collinear points. In Refs. 8 and 9, some thrust controls are derived that require continuously acting propulsion systems, and, in Ref. 10, a variety of different station-keeping techniques are analyzed.

In this paper we investigate the combination of the favorable effects of the gravitational forces with an intermediate thrust application to keep a space probe in a stable orbit around the collinear points. Here we consider only the circular planar restricted three-body problem, but, of course, the sun, eccentricity, etc. would have to be included for real world applications. An optimal control is derived for low-thrust propulsion systems, not so much to stress the optimal way to maneuver, but for the ease with which a workable thrust control can be derived. Numerical simulation results are presented to illustrate the duration of stable motion and show the validity of the linear analysis.

Analysis

The equations of motion for the restricted three-body problem are the familiar ones written in the rotating coordinate system shown in Fig. 1. These equations are (see, for instance, Ref. 11):

$$\ddot{x} - 2\dot{y} = (\partial\Omega/\partial x) + u_x, \quad \ddot{y} + 2\dot{x} = (\partial\Omega/\partial y) + u$$
$$\Omega = \tfrac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + [(1-\mu)/r_1] + (\mu/r_2) \tag{1}$$

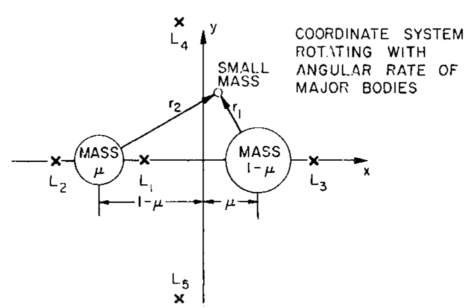

Fig. 1 Coordinate system for the restricted three-body problem and location of libration points.

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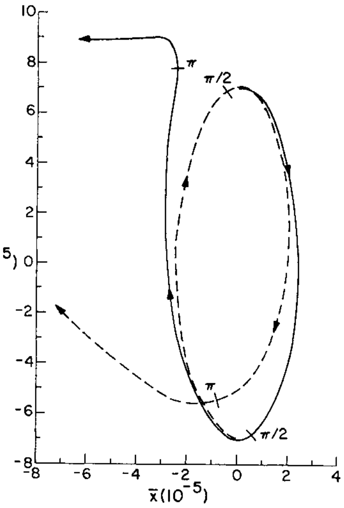

Fig. 2 Free trajectories around L_2 starting on the y axis; Earth-moon system.

The equations are in nondimensional form with the distances nondimensionalized by the distance between the major bodies and time nondimensionalized by the angular rate of rotation of the major bodies. The quantities u_x and u_y are the nondimensional thrust components and μ is the mass ratio of the major bodies ($0 \leq \mu \leq \frac{1}{2}$).

The singular, or libration points ($x = x^{(L)}$, $y = y^{(L)}$; $\dot{x} = \dot{y} = u_x = u_y = 0$) occur at five locations determined by the roots of the equations

$$x^{(L)} - \frac{(1-\mu)[x^{(L)} - \mu]}{[r_1^{(L)}]^3} - \frac{\mu[x^{(L)} + 1 - \mu]}{[r_2^{(L)}]^3} = 0$$

$$y^{(L)}\left\{1 - \frac{(1-\mu)}{[r_1^{(L)}]^3} - \frac{\mu}{[r_2^{(L)}]^3} = 0\right\} \tag{2}$$

The location and identification of these points are illustrated in Fig. 1. If the coordinate system is shifted to any of the libration points and the equations of motion are linearized, the following results are obtained

$$\ddot{\bar{x}} - 2\dot{\bar{y}} = \frac{\partial^2\Omega}{\partial x^2}[x^{(L)},y^{(L)}]\bar{x} + \frac{\partial^2\Omega}{\partial x \partial y}[x^{(L)},y^{(L)}]\bar{y}$$

$$\ddot{\bar{y}} + 2\dot{\bar{x}} = \frac{\partial^2\Omega}{\partial y^2}[x^{(L)},y^{(L)}]\bar{y} + \frac{\partial^2\Omega}{\partial y \partial x}[x^{(L)},y^{(L)}]\bar{x} \tag{3}$$

where $\bar{x} = x - x^{(L)}$ $\bar{y} = y - y^{(L)}$.

For most celestial applications ($\mu < 0.03852$), it is well known that the characteristic roots of the linearized equations indicate that the general infinitesimal motion around the equilateral points L_4 and L_5 is stable, and the motion around the collinear points is unstable. Results have been extended using Liapunov stability analyses to show that these conclusions hold for orbits of finite amplitude as well. For the collinear points, the roots occur as one imaginary pair $\pm i\sigma$ and one real pair $\pm\lambda$. It is possible to pick initial conditions such that the modes corresponding to the real roots do not appear and the linear system will oscillate harmonically. If a trajectory started with these particular initial conditions will stay "stable" for a moderate amount of time, a control can be found that operates intermittently to move the probe back to the proper initial conditions when the nonlinear effects predominate to move the probe on an unstable path. If we apply the following transformation to the linearized Eq. (3), and put the equations into state variable form, we obtain a separated form that is easily identifiable as the conditionally stable and unstable modes of the state trajectory. Let

$$\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{bmatrix} = \begin{bmatrix} c\sigma & 0 & 0 & 1\\ 0 & -\gamma c & \gamma\sigma & 0\\ c_1\lambda & 0 & 0 & -1\\ 0 & -\gamma c_1 & \gamma\lambda & 0\end{bmatrix}\begin{bmatrix}\bar{x}\\ \dot{\bar{x}}\\ \bar{y}\\ \dot{\bar{y}}\end{bmatrix} \tag{4}$$

The equations of motion become

$$\begin{bmatrix}\dot{\xi}_1\\ \dot{\xi}_2\\ \dot{\xi}_3\\ \dot{\xi}_4\end{bmatrix} = \begin{bmatrix} 0 & \lambda & 0 & 0\\ \lambda & 0 & 0 & 0\\ 0 & 0 & 0 & -\sigma\\ 0 & 0 & \sigma & 0\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{bmatrix} + \begin{bmatrix} 0 & 1\\ -\gamma c & 0\\ 0 & -1\\ -\gamma c_1 & 0\end{bmatrix}\begin{bmatrix}u_x\\ u_y\end{bmatrix} \tag{5}$$

The constants are defined as follows:

$$c = 2\sigma/\{\sigma^2 + (\partial^2\Omega/\partial y^2)[x^{(L)},y^{(L)}]\}$$

$$c_1 = -2\lambda/\{\lambda^2 - (\partial^2\Omega/\partial y^2)[x^{(L)},y^{(L)}]\}$$

$$\gamma = -\lambda\sigma/(\partial^2\Omega/\partial x^2)[x^{(L)},y^{(L)}]$$

Values for all constants are given in Table 1 for the Earth-moon system and all three collinear points (taken from Ref. 11).

To obtain pure harmonic motion, we require that the state variables ξ_1 and ξ_2 be zero. This requires the actual positions and velocities to satisfy

$$c\sigma\bar{x}(t) + \dot{\bar{y}}(t) = 0, \; -c\dot{\bar{x}}(t) + \sigma\bar{y}(t) = 0 \tag{6}$$

This set of conditions leads to retrograde elliptical orbits around the collinear points for the linear system. For the Earth-moon system, the orbital eccentricity is about 0.94 for the L_2 point.

To illustrate how long the apparent harmonic motion persists using the exact Eq. (1), some trajectories are plotted in Figs. 2 and 3 for the Earth-moon L_2 point. The initial conditions are such that $\xi_{10} = \xi_{20} = 0$, and a nondimensional distance of 3.5×10^{-5} corresponds to about ten miles. It is clear that one can expect, at most, about one revolution before the body departs from the immediate vicinity of the libration point. Similar results are obtained in Ref. 12 for $\mu = 0.01$ and initial conditions perturbed from $\xi_1 = \xi_2 = 0$.

It now becomes obvious that corrective thrusts might be applied occasionally to re-establish the conditions $\xi_1 = \xi_2 = 0$. Before finding such a control, it is desirable to test for controllability of the linear system. Using the standard criterion,[13] we find that the system is completely controllable (u_x only, u_y only, or both) except when

$$(\partial^2\Omega/\partial y^2)[x^{(L)},y^{(L)}] = 0, 1.5$$

We now formulate the control problem as follows. Find the thrust vector $\mathbf{u}$ to bring the state variables ξ_1 and ξ_2 to zero. The optimal control problem is one of satisfying the above requirements while minimizing fuel consumption, time, etc.

Table 1 Numerical values of constants at the collinear points in the Earth-moon system; μ = 0.0121506683. $\Omega_{xy} = \Omega_{yx} = 0$

Point	L_1	L_2	L_3
Ω_{xx}	11.2951951	7.3808472	3.0213827
Ω_{yy}	−4.1475975	−2.1904236	−0.0106914
λ	2.9320569	2.1586736	0.1778760
σ	2.3343865	1.8626454	1.0104200
c	3.5865005	2.9126036	2.0003223
c_1	−0.4601270	−0.6302425	−8.4039942
γ	−0.6059704	−0.5447672	−0.0594858

For simplicity, we choose for this analysis a low thrust propulsion system and minimize the fuel consumption. The choice of a low thrust system is acceptable because the acceleration in the vicinity of the collinear points is relatively small. This becomes a form of the linear regulator problem when state Eqs. (5) are used. Minimize

$$J = \frac{1}{2}\int_0^{t_f} (u_x^2 + u_y^2)dt$$

subject to Eq. (5) with initial conditions $\xi(0) = \xi_0$ and final conditions $\xi_1(t_f) = \xi_2(t_f) = 0$ and t_f is specified.

This classical problem is easily solved and the following results are obtained:

$$\begin{aligned} u_x(t) &= -\gamma c[u_{xf} \cosh\lambda(t_f - t) + u_{yf} \sinh\lambda(t_f - t)] \\ u_y(t) &= u_{yf} \cosh\lambda(t_f - t) + u_{xf} \sinh\lambda(t_f - t) \end{aligned} \quad (6)$$

where the coefficients are functions of the initial conditions and parameters of the problem

$$u_{xf} = -\left[\frac{\xi_{10}}{2} t_f \sinh\lambda t_f + \frac{\xi_{20}}{2}\left(\frac{1}{\lambda}\sinh\lambda t_f + t_f\cosh\lambda t_f\right)\right] H(u_y)/\Delta + \gamma^2c^2\left[\frac{\xi_{10}}{2} t_f \sinh\lambda t_f - \frac{\xi_{20}}{2}\left(\frac{1}{\lambda}\sinh\lambda t_f - t_f\cosh\lambda t_f\right)\right] H(u_x)/\Delta \quad (7)$$

$$u_{yf} = \left[\frac{\xi_{20}}{2}(t_f \sinh\lambda t_f) - \frac{\xi_{10}}{2}\left(\frac{1}{\lambda}\sinh\lambda t_f - t_f\cosh\lambda t_f\right)\right] H(u_y)/\Delta - \gamma^2c^2\left[\frac{\xi_{10}}{2}\left(\frac{1}{\lambda}\sinh\lambda t_f + t_f\cosh\lambda t_f\right) + \frac{\xi_{20}}{2} t_f \sinh\lambda t_f\right] H(u_x)/\Delta \quad (8)$$

$$\Delta = \frac{1}{4}\left\{\left(\frac{1}{2}\sinh^2\lambda t_f - t_f^2\right)[H(u_y) + \gamma^4c^4H(u_x)] + 2\gamma^2c^2H(u_x)H(u_y)\left(\frac{1}{2}\sinh^2\lambda t_f + t_f^2\right)\right\} \quad (9)$$

The Heaviside step function

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \end{cases}$$

is included so that this result is also applicable for thrust using one component only. The equations for the other state variables, ξ_3 and ξ_4, can easily be integrated to compute the change in the free state variables due to the thrust application.

To more clearly see the structure of the optimal control and the corresponding trajectory change, we make the assumption that the final time t_f is small in comparison with the period of one orbit. This assumption is useful when considering application of the linear control in the nonlinear equations. The application time should be short so that instability effects do not predominate before the control application is complete. When $t_f \ll 2\pi/\lambda$, we can expand the analytic solution in powers of the terminal time with the following result (u_x and u_y both active):

$$u_{xf} \simeq \xi_{10} \times \left[\frac{\gamma^2c^2 - 1}{2\gamma^2c^2} + \frac{(1+\gamma^4c^4)(1-\gamma^2c^2)}{24\gamma^4c^4}\lambda^3t_f^2 + \ldots\right] + \xi_{20}\left[\frac{-1}{\gamma^2c^2t_f} + \frac{(1 - 2\gamma^2c^2 + 3\gamma^4c^4)}{12\gamma^4c^4}\lambda^2t_f + \ldots\right] \quad (10)$$

$$u_{yf} = \xi_{10}\left[\frac{-1}{t_f} + \frac{(3 - 2\gamma^2c^2 + \gamma^4c^4}{12\gamma^2c^2}\lambda^2t_f + \ldots\right] + \xi_{20}\left[\frac{1-\gamma^2c^2}{2\gamma^2c^2}\lambda - \frac{(1-\gamma^2c^2)(1+\gamma^4c^4)}{24\gamma^4c^4}\lambda^3t_f^2 + \ldots\right] \quad (11)$$

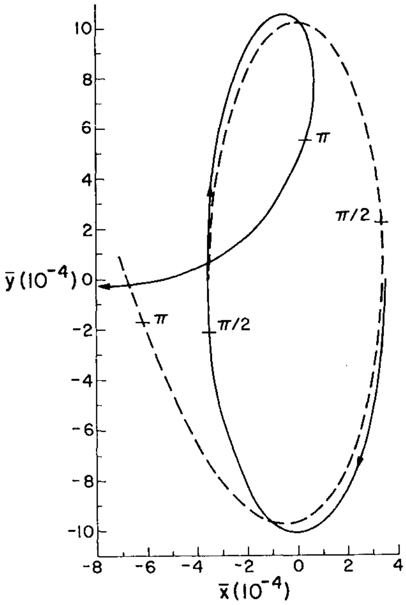

Fig. 3 Free trajectories around L_1 starting on the x axis; Earth-moon system.

The free state variables ξ_3 and ξ_4 become

$$\begin{aligned} \xi_3(t_f) &= \xi_{30}\cos\sigma t_f - \xi_{40}\sin\sigma t_f + \xi_{10} + \left(\frac{\lambda\gamma^2c^2 + \gamma\sigma}{2\gamma^2c^2}\right)t_f\xi_{20} + \ldots \\ \xi_4(t_f) &= \xi_{30}\sin\sigma t_f + \xi_{40}\cos\sigma t_f - \frac{\xi_{20}}{\gamma c^2} + \left(\frac{\sigma\gamma^2c^2 - \lambda\gamma}{2\gamma^2c^2}\right)t_f\xi_{10} + \ldots \end{aligned} \quad (12)$$

For small t_f, the first few terms in the expansion are sufficient to describe the coefficients of the optimal open-loop control. If desired, the time functions $\cosh\lambda(t_f - t)$ and $\sinh\lambda(t_f - t)$ can also be expanded to yield a simple control as a polynomial in time. From Eq. (12), if the control is applied at certain initial conditions,

$$\left|\xi_{10} + \left(\frac{(\lambda\gamma^2c^2 + \gamma\sigma}{2\gamma^2c^2}\right)t_f\xi_{20}\right| \ll \xi_{30},\xi_{40}$$

$$\left|-\frac{1}{\gamma c^2}\xi_{20} + \left(\frac{\sigma\gamma^2c^2 - \lambda\gamma}{2\gamma^2c^2}\right)t_f\xi_{10}\right| \ll \xi_{30},\xi_{40}$$

hardly any change will occur in the shape of the original elliptic trajectory. This fact is substantiated in the simulation results. The performance index J can also be evaluated for small terminal time,

$$J = \frac{1}{t_f}\left(\frac{\xi_{10}^2}{2} + \frac{\xi_{20}^2}{2\gamma^2c^2}\right) + \frac{\xi_{10}\xi_{20}}{2}\left[\frac{\lambda(1+\gamma^2c^2)}{\gamma^2c^2}\right] + 0(t_f) \quad (13)$$

The same analysis for radial or cross-axis control only yields a leading term of order $1/t_f^3$, consequently, these one-dimensional thrust actions will require more fuel expenditure. Equation (13) can be used to estimate long-time (nondimensional) fuel consumption if the control is applied at approximately the same initial conditions on each orbit.

Results and Conclusions

A numerical simulation has been performed by applying the optimal control described above to the nonlinear equations of motion. The trajectory chosen for illustration is

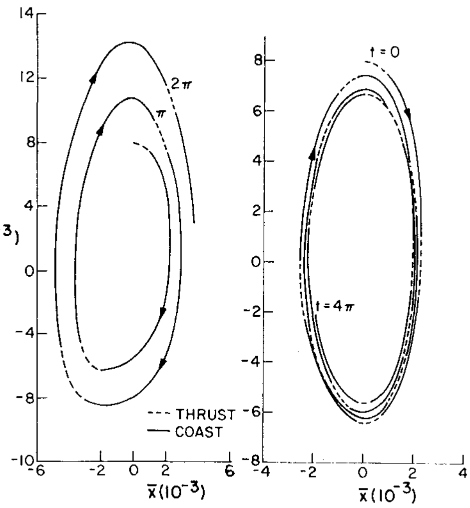

Fig. 4 Trajectories around L_2 with corrective thrust applications every $\pi/2$ rad (left) and every $\pi/4$ rad (right).

around the L_2 point in the Earth-moon system and is described in Ref. 9. The maximum amplitude is such that the satellite can stay partially visible from Earth when out-of-plane motion is also considered. For this simulation the corrective thrust is applied at fixed time intervals for a short duration. The changes in the predicted elliptic trajectory are sensitive to both the time and place where the control is applied. If relatively long time intervals are taken between corrections, the trajectory will change after every application and stability cannot be maintained. These results are shown in Fig. 4. The time intervals correspond to thrust applications every seven days with a duration of about one day. If we make the interval between corrections smaller, we can establish a bounded quasi-elliptic trajectory for an extended period of time. This is shown in Fig. 4 for the same initial conditions, but the interval between applications is now about 3.5 days. For orbits much closer to the libration point (Fig. 5), the interval can be lengthened because the nonlinear instability effects do not occur as rapidly; hence, the interval between corrections is dependent on the amplitude of the orbit. In this case there is a maximum deviation of 10% between the first and fourth orbits. In the last two figures, the simulation is carried out for a period of two revolutions of the major bodies.

The simulation results indicate that it is possible to establish quasi-elliptic orbits around the collinear points for the restricted three-body model combining the favorable effects of the gravitational forces with short corrective thrust applications. Furthermore, the magnitude of the corrections for the low thrust propulsion system used here is within the range of present systems, although impulsive thrusts would be equally acceptable to accomplish the corrections. Another interesting observation concerns the applicability of the linear control in the actual system. Usually, the linear results can serve as no more than an estimate or guide for the actual control of a nonlinear system, and refined techniques are still necessary for accurate response. In this case, however, if the linear control is applied at frequent intervals, it does establish positions and velocities accurate enough to maintain limited stability. This same idea is also applicable to the stable equilateral points L_4 and L_5. In this case the stable motion is a combination of two harmonic modes of long and short duration. A similar analysis can be made to find corrective thrusts to suppress one mode and allow the body to oscillate in the other. Compared to continuous feedback control systems, we can expect less fuel consumption for this station-keeping maneuver. For instance, the cost of maintaining the orbit shown in Fig. 5 is about 5 fps/yr. The only restriction is that the probe must nominally stay in the particular elliptic orbits specified by the linearized analysis.

The results presented here are, of course, restricted by the model assumed. Analyses should be made to determine the perturbation effects of the sun, ellipticity of the Earth-moon system, etc., on the control method analyzed here.

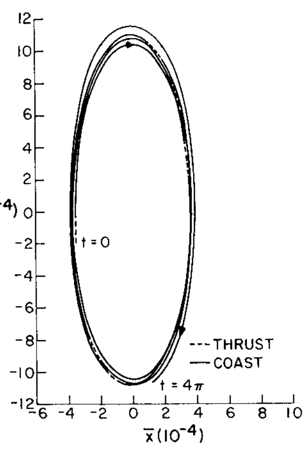

Fig. 5 Trajectory around L_2 with corrective thrust every $\pi/2$ rad.

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